Li–Yau Inequalities for Dunkl Heat Equations

黎怀谦 (Li, Huaiqian)

Center for Applied Mathematics
Tianjin University
Tianjin 300072, P. R. China

(joint work with Qian, Bin)

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Outline

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$(\mathbb{M}^n, \rho, |\cdot|, \Delta, \nabla)$

Li–Yau [Acta Math. 1986]: Assume Ric ≥ 0. Then for every positive solution to the heat equation $\partial_t u = \Delta u$ on $(0,\infty) \times \mathbb{M}$,

$$-\Delta\big(\log u(t,\cdot)\big)(x) \leq \frac{n}{2t}, \quad t > 0,\ x \in \mathbb{M},$$

and equivalently,

$$\frac{|\nabla u(t,\cdot)(x)|^2}{u(t,x)^2} - \frac{\partial_t u(t,x)}{u(t,x)} \leq \frac{n}{2t}, \quad t > 0,\ x \in \mathbb{M},$$

which implies the Harnack inequality

$$u(s,x) \leq u(t,y)\left(\frac{t}{s}\right)^{n/2} \exp\left(\frac{\rho(x,y)^2}{4(t-s)}\right), \quad 0 < s < t < \infty,\ x,y \in \mathbb{M}.$$

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Basic notions

Consider the Euclidean space $\mathbb{R}^d$ with the standard scalar product $\langle \cdot, \cdot \rangle$ and the induced norm $|\cdot| = \sqrt{\langle \cdot, \cdot \rangle}$.

- Reflection r_α: for $\alpha \in \mathbb{R}^d \setminus \{0\}$,

$$r_\alpha x = x - 2\frac{\langle \alpha, x \rangle}{|\alpha|^2}\alpha, \quad x \in \mathbb{R}^d,$$

  which is a reflection in the hyperplane $\alpha^\perp$.

- Root system $\mathfrak{R}$: finite set in $\mathbb{R}^d \setminus \{0\}$ such that $\forall\, \alpha \in \mathfrak{R}$,

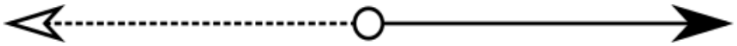
$$r_\alpha(\mathfrak{R}) = \mathfrak{R} \quad \text{and} \quad \mathfrak{R} \cap \alpha\mathbb{R} = \{\alpha, -\alpha\}.$$

  Normalize $|\alpha| = \sqrt{2}$, $\alpha \in \mathfrak{R}$.

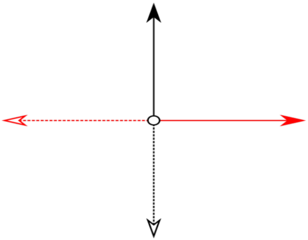
- Reflection group G: finite group generated by $\{r_\alpha : \alpha \in \mathfrak{R}\}$.
- Multiplicity function κ: G-invariant map $\kappa_\cdot : \mathfrak{R} \to \mathbb{R}_+$, i.e.,

$$\kappa_{g\alpha} = \kappa_\alpha, \quad g \in G,\ \alpha \in \mathfrak{R}.$$

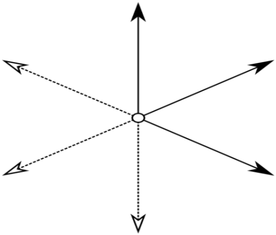
Examples of root systems



A_1



$A_1 \times A_1$



A_2

Dunkl operator

Let $\mathfrak{R}_+$ be the positive subsystem such that $\mathfrak{R} = \mathfrak{R}_+ \uplus (-\mathfrak{R}_+)$.

Definition (C.F. Dunkl: Trans. AMS 1989)

Given a root system $\mathfrak{R}$ and a multiplicity function $\kappa. : \mathfrak{R} \to \mathbb{R}_+$, for every $\xi \in \mathbb{R}^d$, the Dunkl operator along ξ is defined by

$$\mathrm{D}_\xi f(x) = \partial_\xi f(x) + \sum_{\alpha \in \mathfrak{R}_+} \kappa_\alpha \langle \alpha, \xi \rangle \frac{f(x) - f(r_\alpha x)}{\langle \alpha, x \rangle}, \quad f \in C^1(\mathbb{R}^d),\ x \in \mathbb{R}^d,$$

where ∂_ξ denotes the directional derivative along ξ.

Note that $\mathrm{D}_\xi \circ \mathrm{D}_\eta = \mathrm{D}_\eta \circ \mathrm{D}_\xi$, $\xi, \eta \in \mathbb{R}^d$, and

$$\frac{f(x) - f(r_\alpha x)}{\langle \alpha, x \rangle} = -\frac{1}{\langle \alpha, x \rangle} \int_0^1 \frac{\partial}{\partial t} f\big(x - t\langle \alpha, x \rangle \alpha\big)\, \mathrm{d}t = \int_0^1 \partial_\alpha f\big(x - t\langle \alpha, x \rangle \alpha\big)\, \mathrm{d}t.$$

However, **no** classic Leibniz and chain rules in general!

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Dunkl gradient/Laplacian

Let $\{e_j : j = 1, \cdots, d\}$ be the standard orthonormal basis of $\mathbb{R}^d$ and let $D_j = D_{e_j}$. The Dunkl gradient operator and the Dunkl Laplacian are respectively defined as

$$\nabla_\kappa = (D_1, \cdots, D_d), \quad \Delta_\kappa = \sum_{j=1}^{d} D_j^2.$$

More precisely, $\forall f \in C^2(\mathbb{R}^d)$,

$$\Delta_\kappa f(x) = \Delta f(x) + 2 \sum_{\alpha \in \mathfrak{R}_+} \kappa_\alpha \left(\frac{\langle \alpha, \nabla f(x) \rangle}{\langle \alpha, x \rangle} - \frac{f(x) - f(r_\alpha x)}{\langle \alpha, x \rangle^2} \right), \quad x \in \mathbb{R}^d.$$

In particular, when $\kappa = 0$, $\nabla_0 = \nabla$ and $\Delta_0 = \Delta$.

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Typical example 1

Example (rank-one case)

Let $d = 1$. Then $\mathfrak{R} = \{-\sqrt{2}, \sqrt{2}\}$, $G = \{e, r\}$ with $e(x) = x$ and $r(x) = -x$ for every $x \in \mathbb{R}$. Given $\lambda \in \mathbb{R}_+$, the Dunkl operator is

$$\mathrm{D}f(x) = f'(x) + \lambda \frac{f(x) - f(-x)}{x}, \quad x \in \mathbb{R}, f \in C^1(\mathbb{R}),$$

and the Dunkl Laplacian is

$$\mathrm{D}^2 f(x) = f''(x) + \frac{\lambda}{x^2}\left[f(-x) - f(x) + 2xf'(x)\right], \quad x \in \mathbb{R}, f \in C^2(\mathbb{R}).$$

Typical example 2

Example (radial Dunkl process)

Let $W = \{x \in \mathbb{R}^d : \langle \alpha, x\rangle > 0,\ \alpha \in \mathfrak{R}_+\}$ and let $\overline{W}$ be its closure. The *radial Dunkl process* is defined as the $\overline{W}$-valued Markov process with infinitesimal generator

$$\Delta_\kappa^W f(x) = \frac{1}{2}\Delta f(x) + \sum_{\alpha \in \mathfrak{R}_+} \kappa_\alpha \frac{\langle \alpha, \nabla f(x)\rangle}{\langle \alpha, x\rangle},$$

where $f \in C^2(\overline{W})$ such that $\langle \alpha, \nabla f(x)\rangle = 0$ whenever $\langle \alpha, x\rangle = 0$. It is known that the corresponding SDE, i.e.,

$$\mathrm{d}Y_t = \mathrm{d}B_t + \sum_{\alpha \in \mathfrak{R}_+} \kappa_\alpha \frac{\mathrm{d}t}{\langle \alpha, Y_t\rangle}, \quad Y_0 = y \in \overline{W},$$

has a unique strong solution for all $t \geq 0$ (see e.g. O. Chybiryakov [SPA 2006], B. Schapira [PTRF 2007]).

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A remark on Dunkl process

$$\big(\Delta_\kappa, \mathcal{D}(\Delta_\kappa)\big) \quad \Longleftrightarrow \quad \text{Dunkl process } (X_t)_{t\geq 0}$$

$\forall\, I \subset \mathfrak{R}_+$, let

$$U_I = \{\alpha \in \mathfrak{R}_+ : \langle \alpha, x\rangle = 0,\ x \in \cap_{\alpha\in I}\alpha^\perp\}.$$

$(X_t)_{t\geq 0}$ is a càdlàg Markov process of jump type with jumping kernel

$$K(x, dy) = \begin{cases} \sum_{\alpha\in\mathfrak{R}_+} \frac{2\kappa_\alpha}{\langle\alpha,x\rangle^2}\delta_{r_\alpha x}(\mathrm{d}y), & x \in \mathbb{R}^d \setminus (\cup_{\alpha\in\mathfrak{R}_+}\alpha^\perp), \\ \sum_{\alpha\in\mathfrak{R}_+\setminus U_I} \frac{2\kappa_\alpha}{\langle\alpha,x\rangle^2}\delta_{r_\alpha x}(\mathrm{d}y), & x \in \cap_{\alpha\in I}\alpha^\perp, \\ 0, & x = 0, \end{cases}$$

where I is any subset of $\mathfrak{R}_+$, $\delta_\cdot$ denotes the Dirac measure.

M. Rösler and M. Voit [Adv. App. Math. 1998], L. Gallardo and M. Yor [PTRF 2005] & [AOP 2006], B. Schapira [PTRF 2007], etc.

Weight function

Let

$$\omega_\kappa(x) := \prod_{\alpha \in \mathfrak{R}_+} |\langle \alpha, x \rangle|^{\kappa_\alpha}, \quad x \in \mathbb{R}^d,$$

which is a homogeneous function of degree

$$\lambda_\kappa := \sum_{\alpha \in \mathfrak{R}_+} \kappa_\alpha.$$

Let

$$\mu_\kappa(\mathrm{d}x) := \omega_\kappa(x)\mathrm{d}x,$$

where $\mathrm{d}x$ denotes the Lebesgue measure on $\mathbb{R}^d$.

Dunkl heat semigroup/kernel

Let $(P_t)_{t>0}$ be the Dunkl semigroup with $P_t = e^{t\Delta_\kappa}$ for every $t > 0$. Then P_t admits the Dunkl heat kernel $p_t(x, y)$ w.r.t. μ_κ, which is a C^∞ function of all variables $x, y \in \mathbb{R}^d$ and $t > 0$, and satisfies that

$$\partial_t p_t(x,y) = \Delta_\kappa p_t(\cdot, y)(x), \ p_t(x,y) = p_t(y,x) > 0, \ \int_{\mathbb{R}^d} p_t(x,y)\, \mathrm{d}\mu_\kappa(y) = 1;$$

moreover,

$$p_t(x,y) \le \frac{1}{c_\kappa (2t)^{d/2+\lambda_\kappa}} \exp\Big(-\frac{\delta(x,y)^2}{4t}\Big), \quad x, y \in \mathbb{R}^d, \ t > 0,$$

where $c_\kappa := \int_{\mathbb{R}^d} e^{-|x|^2/2}\, \mu_\kappa(\mathrm{d}x)$ and $\delta(x,y) := \min_{g\in G} |gx - y|$.

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Li–Yau inequalities for Dunkl heat equation

Theorem (L.–Qian 2021)

Let $T \in (0, \infty]$ *and* $\beta : (0, T) \times \mathbb{R}^d \to \mathbb{R}$ *be a function. Suppose that* $u : [0, T) \times \mathbb{R} \to (0, \infty)$ *is any* C^2 *solution to the Dunkl heat equation*

$$\partial_t u(t, x) = \Delta_\kappa\big(u(t, \cdot)\big)(x), \quad (t, x) \in (0, T) \times \mathbb{R}^d.$$

Then

$$-\Delta_\kappa\big(\log p_t(\cdot, y)\big)(x) \leq \beta(t, x), \quad (t, x, y) \in (0, T) \times \mathbb{R}^d \times \mathbb{R}^d, \tag{1}$$

is equivalent to

$$-\Delta_\kappa\big(\log u(t, \cdot)\big)(x) \leq \beta(t, x), \quad (t, x) \in (0, T) \times \mathbb{R}^d; \tag{2}$$

moreover, either (1) *or* (2) *implies*

$$\frac{|\nabla u(t, \cdot)(x)|^2}{u(t, x)^2} - \frac{\partial_t u(t, x)}{u(t, x)} \leq \beta(t, x), \quad (t, x) \in (0, T) \times \mathbb{R}^d.$$

Remarks

The idea of proof is motivated by C. Yu and F. Zhao [JGA, 2020], where sharp Li–Yau inequalities for the Laplace–Beltrami operator on hyperbolic spaces were obtained.

Recently, similar idea was employed by F. Weber and R. Zacher in [arXiv:2012.12974] to prove Li–Yau inequalities for the fractional Laplacian $(-\Delta)^s$ with $s \in (0, 1)$.

Outline

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Sharp Li–Yau inequalities for Dunkl heat kernel

Theorem (L.–Qian 2021)

Suppose that G is isomorphic to $\mathbb{Z}_2^d = \{0,1\}^d$. Then

$$-\Delta_\kappa\big(\log p_t(\cdot,y)\big)(x) \le \frac{d+2\lambda_\kappa}{2t}, \quad x,y\in\mathbb{R}^d,\ t>0.$$

If $\kappa = 0$, then $\Delta_\kappa = \Delta$, $\lambda_\kappa = 0$, and $(p_t)_{t>0}$ is the heat kernel associated to Δ on $\mathbb{R}^d$, i.e.,

$$p_t(x,y) = \frac{1}{(4\pi t)^{d/2}}\exp\Big(-\frac{|x-y|^2}{4t}\Big), \quad x,y\in\mathbb{R}^d,\ t>0;$$

hence

$$-\Delta\big(\log p_t(\cdot,y)\big)(x) = \frac{d}{2t}, \quad x,y\in\mathbb{R}^d,\ t>0.$$

Dunkl heat kernel: $\mathbb{Z}_2^d$ case

Let $\Gamma(\cdot)$ be the Gamma function. For every $t > 0$, $u, v \in \mathbb{R}$ and each $i = 1, \cdots, d$, let

$$p_t^i(u,v) = \frac{1}{c_{\kappa_i}(2t)^{\kappa_i+1/2}} \exp\Big(-\frac{u^2+v^2}{4t}\Big) E_{\kappa_i}\Big(\frac{u}{\sqrt{2t}}, \frac{v}{\sqrt{2t}}\Big),$$

where $c_{\kappa_i} := \Gamma(\kappa_i + 1/2)$ and

$$E_{\kappa_i}(u,v) := \frac{\Gamma(\kappa_i+1/2)}{\Gamma(1/2)\Gamma(\kappa_i)} \int_{-1}^{1} (1-s)^{\kappa_i-1}(1+s)^{\kappa_i} e^{suv}\,\mathrm{d}s.$$

Then, for every $t > 0$ and every $x, y \in \mathbb{R}^d$ with $x = (x_1, \cdots, x_d)$ and $y = (y_1, \cdots, y_d)$,

$$p_t(x,y) = \prod_{i=1}^{d} p_t^i(x_i, y_i).$$

Sketched proofs (I)

Let $t > 0$ and $x, y \in \mathbb{R}^d$ with $x = (x_1, \cdots, x_d), y = (y_1, \cdots, y_d)$. Then

$$\begin{aligned}
&\Delta_\kappa\big(\log p_t(\cdot, y)\big)(x) \\
&= \sum_{j=1}^{d} \Big(\frac{\kappa_j}{x_j^2} \big[2x_j \partial_{x_j} \log p_t^j(x_j, y_j) - \log p_t^j(x_j, y_j) + \log p_t^j(-x_j, y_j) \big] \\
&\quad + \partial^2_{x_j x_j} \log p_t^j(x_j, y_j) \Big).
\end{aligned}$$

Hence, it suffices to estimate the terms in the parentheses, i.e.,

$$\frac{\kappa_j}{x_j^2} \big[2x_j \partial_{x_j} \log p_t^j(x_j, y_j) - \log p_t^j(x_j, y_j) + \log p_t^j(-x_j, y_j) \big] + \partial^2_{x_j x_j} \log p_t^j(x_j, y_j),$$

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Sketched proofs (II)

Ignoring the subscript j, for every $t > 0$ and $x, y \in \mathbb{R}$, we let

$$\mathrm{I} := \partial_{xx}^2 \log p_t(x,y) + \frac{\kappa}{x^2}\left[2x\partial_x \log p_t(x,y) - \log p_t(x,y) + \log p_t(-x,y)\right],$$

where

$$p_t(x,y) = \frac{1}{c_\kappa (2t)^{\kappa+1/2}} \exp\left(-\frac{x^2+y^2}{4t}\right) E_\kappa\left(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\right),$$

$$E_\kappa\left(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\right) = C_\kappa \int_{-1}^{1} (1-s)^{\kappa-1}(1+s)^\kappa e^{\frac{sxy}{2t}}\, \mathrm{d}s.$$

Then we only need to estimate

$$\mathrm{I}_1 := \partial_{xx}^2 \log p_t(x,y),$$

$$\mathrm{I}_2 := \frac{\kappa}{x^2}\left[2x\partial_x \log p_t(x,y) - \log p_t(x,y) + \log p_t(-x,y)\right].$$

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Sketched proofs (III)

Let $g(s) = (1-s)^{\kappa-1}(1+s)^{\kappa}$ and $a = xy/(2t)$.

By the Cauchy–Schwarz inequality,

$$\mathrm{I}_1 = -\frac{1}{2t} + \frac{y^2}{4t^2}\left(\frac{\int_{-1}^{1} s^2 g(s)e^{as}\,\mathrm{d}s}{\int_{-1}^{1} g(s)e^{as}\,\mathrm{d}s} - \frac{\left(\int_{-1}^{1} sg(s)e^{as}\,\mathrm{d}s\right)^2}{\left(\int_{-1}^{1} g(s)e^{as}\,\mathrm{d}s\right)^2}\right)$$

$$\geq -\frac{1}{2t}.$$

$$\mathrm{I}_2 = \frac{\kappa}{x^2}\left[-\frac{x^2}{t} + 2a\frac{\int_{-1}^{1} sg(s)e^{as}\,\mathrm{d}s}{\int_{-1}^{1} g(s)e^{as}\,\mathrm{d}s} + \log\frac{\int_{-1}^{1} g(s)e^{-as}\,\mathrm{d}s}{\int_{-1}^{1} g(s)e^{as}\,\mathrm{d}s}\right]$$

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Thus, $\mathrm{I} = \mathrm{I}_1 + \mathrm{I}_2 \geq -(1+2\kappa)/(2t)$.

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An immediate consequence of the Li–Yau inequality is the Harnack inequality.

Corollary (L.–Qian 2021)

Let $T \in (0, \infty]$. Suppose that G is isomorphic to $\mathbb{Z}_2^d$ and $u : [0, T) \times \mathbb{R} \to (0, \infty)$ is a C^2 solution to the Dunkl heat equation, i.e.,

$$\partial_t u(t, x) = \Delta_\kappa \big(u(t, \cdot)\big)(x), \quad (t, x) \in (0, T) \times \mathbb{R}^d.$$

Then, for every $0 < s < t < T$ and every $x, y \in \mathbb{R}^d$,

$$u(s, x) \leq u(t, y) \Big(\frac{t}{s}\Big)^{\lambda_\kappa + d/2} \exp\Big(\frac{|x - y|^2}{4(t - s)}\Big).$$

Outline

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Li–Yau inequalities of "gradient" type

Let $f, g \in C^2(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. Then

$$\Gamma(f,g)(x) := \frac{1}{2}\left[\Delta_\kappa(fg) - f\Delta_k g - g\Delta_\kappa f\right](x)$$
$$= \langle \nabla f(x), \nabla g(x)\rangle + \sum_{\alpha\in\mathfrak{R}_+} \kappa_\alpha \frac{\big(f(x)-f(r_\alpha x)\big)\big(g(x)-g(r_\alpha x)\big)}{\langle \alpha, x\rangle^2}.$$

Set $\Gamma(f) = \Gamma(f,f)$ for convenience.

Q: Does any of the following hold, i.e.,

$$\frac{\Gamma\big(p_t(\cdot,y)\big)(x)}{p_t(x,y)^2} - \frac{\partial_t p_t(x,y)}{p_t(x,y)} \le \frac{d+2\lambda_\kappa}{2t}$$

and

$$\frac{|\nabla_\kappa p_t(\cdot,y)(x)|^2}{p_t(x,y)^2} - \frac{\partial_t p_t(x,y)}{p_t(x,y)} \le \frac{d+2\lambda_\kappa}{2t} \,?$$

" Thank you everyone! "